DE GRUYTER DOI 10.1515/advgeom-2015-0010 — Adv. Geom. 2015; 15 (2):241–244

Łukasz Kosiński, Tomasz Warszawski and Włodzimierz Zwonek

Geometric properties of semitube domains

Abstract: We study the geometry of semitube domains in $\mathbb{C}^2$, in particular we extend the result of Burgués and Dwilewicz for semitube domains by dropping the smoothness assumption. We also prove various properties of non-smooth pseudoconvex semitube domains, obtaining a relation between pseudoconvexity of a semitube domain and the number of components of its vertical slices. Finally, we present an example of a non-convex domain in $\mathbb{C}^n$ such that its image under arbitrary isometries is pseudoconvex.

Keywords: Semitube domains, Hartogs–Laurent domains, Bochner's theorem, multisubharmonic functions.

2010 Mathematics Subject Classification: Primary 32V20; Secondary 32L05, 32V25

Łukasz Kosiński, Tomasz Warszawski, Włodzimierz Zwonek: Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland,
email: Lukasz.Kosinski@im.uj.edu.pl, Tomasz.Warszawski@im.uj.edu.pl, Wlodzimierz.Zwonek@im.uj.edu.pl

Communicated by: G. Gentili

1 Introduction

A theorem of Bochner states that a tube domain in $\mathbb{C}^n$ is pseudoconvex if and only if it is convex. This fact is a starting point for our considerations. In [1] a similar problem was considered for semitube domains — domains that are invariant in one real direction (they were considered in $\mathbb{C}^2$). Formally the *semitube domain* (*set*) *with the base B* being a domain (set) lying in $\mathbb{R}^3$ is defined as follows

$$S_B := \{z \in \mathbb{C}^2 : (z_1, \operatorname{Re} z_2) \in B\},$$

which may be rewritten as $B \times \mathbb{R}$. We observe that there is no direct analogue of the Bochner theorem in the class of semitube domains; this follows easily from the fact that any domain $D \subset \mathbb{C}$ induces a pseudoconvex domain of the form $S_{D\times(0,1)}$. However, it was recently proven by Burgués and Dwilewicz that some additional requirement implies the convexity of a semitube domain. Namely, the main result of [1] is that under the additional assumption of smoothness any domain $D \subset \mathbb{R}^3$ such that for any isometry A of $\mathbb{R}^3$ the semitube domain $S_{A(D)} = A(D) \times \mathbb{R}$ is pseudoconvex must be convex. The main aim of our paper is to prove this result without the smoothness assumption. The methods used in our paper are also quite different.

Theorem 1. *Let $D \subset \mathbb{R}^3$ be a domain such that the semitube $S_{A(D)}$ is pseudoconvex for any isometry A of $\mathbb{R}^3$. Then D is convex.*

Another natural question that arises while considering semitube domains is the problem whether one can exhaust any pseudoconvex semitube domain with smooth semitube domains. This is the case:

Theorem 2. *Any pseudoconvex semitube domain $G \subset \mathbb{C}^2$ can be exhausted by $\mathcal{C}^\infty$-smooth strongly pseudoconvex semitube domains.*

The mapping $\pi : \mathbb{C}^2 \ni z \longmapsto (z_1, \exp(z_2)) \in \mathbb{C}^2$ induces a holomorphic covering between semitube domains S_D and Hartogs–Laurent domains $\pi(S_D)$. We call a domain $G \subset \mathbb{C}^2$ a *Hartogs–Laurent domain* if any non-empty fiber $\{z_2 \in \mathbb{C} : (z_1, z_2) \in G\}$ is some union of annuli, i.e. sets of the form $\{z_2 \in \mathbb{C} : r < |z_2| < R\}$ with $0 \le r < R \le \infty$. The projection of G on the first coordinate is called the *base* of the domain. The mapping π induces a one-to-one correspondence between those two classes of domains as stated in the following proposition.

Proposition 3. *Let π be as above. Then the function $S_D \longmapsto \pi(S_D)$ is a one-to-one correspondence between the class of all pseudoconvex semitube domains in $\mathbb{C}^2$ and the class of all pseudoconvex Hartogs–Laurent domains in $\mathbb{C}^2$.*

Proof. Let the domain S_D be pseudoconvex. Then $u := -\log d_{S_D} \in \mathrm{PSH}(S_D)$, where d_G is the distance to the boundary of G. Since u does not depend on $\operatorname{Im} z_2$, the function v given by the formula $v(z) := u(z_1, \log z_2)$, $z \in \pi(S_D)$, is well-defined and plurisubharmonic on $\pi(S_D)$. Therefore,

$$\widetilde{v}(z) := \max\{v(z), \|z\|, -\log|z_2|\}, \quad z \in \pi(S_D),$$

is an exhaustion plurisubharmonic function for $\pi(S_D)$. The other implication is trivial. □

The above observation shows that there is a very natural relation between (pseudoconvex) semitube domains and (pseudoconvex) Hartogs–Laurent domains. There is a very rich literature on that class of domains (see e.g. [5]) which shows that many properties of pseudoconvex semitube domains may be obtained from the properties of pseudoconvex Hartogs–Laurent domains. In particular, very irregular Hartogs–Laurent domains (like the worm domains in [2]) produce very irregular pseudoconvex semitube domains.

2 Proofs of Theorem 1 and Theorem 2

Proof of Theorem 1. Suppose that D is not convex. The idea of the proof is the following. We find a sequence of parallel segments of constant length lying in the domain D and such that the limit segment I intersects the boundary at some inner point, whereas the boundary of the limit segment lies in the domain. Then we rotate the domain D so that I becomes parallel to the $\operatorname{Re} z_2$ axis. The image of the rotated semitube domain under π is a pseudoconvex Hartogs–Laurent domain with a sequence of annuli lying in the domain. The pseudoconvexity of the Hartogs–Laurent domain leads to a contradiction with the Kontinuitätssatz.

Let us proceed now formally. By [3, Theorem 2.1.27] there is a point $a \in \partial D$ and a quadratic polynomial P on $\mathbb{R}^3$ such that

- $P(a) = 0$ and $v := \nabla P(a) \neq 0$;
- $\langle v, X\rangle = 0$ and $C := -\mathcal{H}P(a; X) > 0$ for some $X \in \mathbb{R}^3$;
- $P(x) < 0$ implies $x \in D$ for $x \in \mathbb{R}^3$ near a.

By ∇ and $\mathcal{H}$ we denoted the gradient and the Hessian. One may assume that $\|v\| = 1$.

For $\varepsilon \geq 0$ and $\delta \in \mathbb{R}$ such that $(\varepsilon, \delta) \neq (0,0)$, $\varepsilon\mathcal{H}P(a; v) \leq 1$ and $4|\delta v^T \mathcal{H}P(a)X| \leq 1$, we have

$$\begin{aligned} P(a - \varepsilon v + \delta X) &= P(a) + \langle \nabla P(a), -\varepsilon v + \delta X\rangle + \frac{1}{2}\mathcal{H}P(a; -\varepsilon v + \delta X) \\ &= -\varepsilon + \frac{1}{2}\mathcal{H}P(a; -\varepsilon v) + \frac{1}{2}\mathcal{H}P(a; \delta X) - \varepsilon\delta v^T\mathcal{H}P(a)X \\ &\leq -\varepsilon + \frac{1}{2}\varepsilon^2\mathcal{H}P(a; v) - \frac{1}{2}C\delta^2 + \frac{1}{4}\varepsilon \leq -\frac{1}{2}\varepsilon - \frac{1}{2}C\delta^2 + \frac{1}{4}\varepsilon < 0. \end{aligned}$$

It means that $a - \varepsilon v + \delta X \in D$ if this point is sufficiently close to a (i.e. if (ε, δ) is sufficiently close to $(0,0)$ but not equal to $(0,0)$ and $\varepsilon \geq 0$). In particular, there exists a closed non-degenerate rectangle $R \subset \mathbb{R}^3$ such that $a \in \partial R \cap \partial D$, the point a is not a vertex of R and $R \setminus \{a\} \subset D$.

There is an isometry A such that $A(R) = [\alpha, \beta] \times \{0\} \times [\alpha', \beta'] \subset \mathbb{R}^3$ and $A(a) \in \{\alpha, \beta\} \times \{0\} \times (\alpha', \beta')$ for some real numbers $\alpha < \beta$ and $\alpha' < \beta'$; without loss of generality we assume that $A(a) \in \{(\beta, 0)\} \times (\alpha', \beta')$. Recall that $S_{A(D)}$ is pseudoconvex. Recall also that the Hartogs–Laurent domain $\Omega := \pi(S_{A(D)}) \subset \mathbb{C}^2$ is pseudoconvex; because of the form of $A(D)$ we get a family of holomorphic mappings

$$f_b(\lambda) := (b, \lambda), \quad \lambda \in \overline{\mathbb{A}}(e^{\alpha'}, e^{\beta'}),\ b \in [\alpha, \beta], \ \text{ where } \mathbb{A}(p,q) := \{\lambda \in \mathbb{C} : p < |\lambda| < q\},$$

such that

$$\bigcup_{b \in [\alpha, \beta)} f_b(\overline{\mathbb{A}}(e^{\alpha'}, e^{\beta'})) \subset \Omega \quad \text{and} \quad \bigcup_{b \in [\alpha, \beta]} f_b(\partial\mathbb{A}(e^{\alpha'}, e^{\beta'})) \subset\subset \Omega.$$

However, $f_\beta(\overline{\mathbb{A}}(e^{\alpha'}, e^{\beta'})) \not\subset \Omega$, which contradicts the Kontinuitätssatz as formulated in [4, Theorem 4.1.19]. □

Proof of Theorem 2. Let $u := -\log d_G \in \mathrm{PSH}(G)$ and $G_\varepsilon := \{z \in G : d_G(z) > \varepsilon\}$ for $\varepsilon \in (0, 1)$. Define the standard regularisations u_ε of u with the help of convolution with radial functions. We have $u_\varepsilon \in \mathrm{PSH} \cap \mathcal{C}^\infty(G_\varepsilon)$ and $u_\varepsilon \searrow u$ if $\varepsilon \searrow 0$. Moreover, u_ε does not depend on $\operatorname{Im} z_2$.

For $\varepsilon \in (0, 1)$ and $\delta > 0$ define

$$\tilde{u}_\varepsilon(z) := u_\varepsilon(z) + \varepsilon\|(z_1, \operatorname{Re} z_2)\|^2, \quad \widetilde{G}_{\varepsilon,\delta} := \{z \in G_\varepsilon : \tilde{u}_\varepsilon(z) < 1/\delta\}.$$

Note that $\overline{\widetilde{G}_{\varepsilon,\delta}} \subset G_\varepsilon$ for $\delta > -1/\log\varepsilon$. Indeed, if $z_n \in \widetilde{G}_{\varepsilon,\delta}$, $z_n \to z$, then $u(z_n) \le \tilde{u}_\varepsilon(z_n) < 1/\delta < -\log\varepsilon$, so $u(z) < -\log\varepsilon$.

By the Sard Theorem for every $\varepsilon > 0$ the set A_ε of all real numbers $\delta > 0$ such that $\nabla\tilde{u}_\varepsilon(z) \neq 0$ if $\tilde{u}_\varepsilon(z) = 1/\delta$ is dense in $\mathbb{R}_+$. For $n \in \mathbb{N}$ we choose a number $\delta_{1/n}$ such that $\delta_{1/n} > -1/\log(1/n)$ and $\delta_{1/n} \in A_{1/n}$. Since the minorants $-1/\log(1/n)$ tend to zero, we may assume additionally that $\delta_{1/n} \searrow 0$ as $n \nearrow \infty$. Then we define

$$\widetilde{G}_{1/n} := \widetilde{G}_{1/n,\delta_{1/n}}.$$

The following properties

- $\tilde{u}_{1/n} - 1/\delta_{1/n}$ are $\mathcal{C}^\infty$-smooth strongly plurisubharmonic defining functions of $\widetilde{G}_{1/n}$
- $\tilde{u}_\varepsilon$ are independent on $\operatorname{Im} z_2$

imply that the sets $\widetilde{G}_{1/n}$ are open $\mathcal{C}^\infty$-smooth strongly pseudoconvex semitube sets. We directly check that $\widetilde{G}_{1/n} \subset \widetilde{G}_{1/m} \subset G$ if $n < m$ and that every $z \in G$ belongs to some $\widetilde{G}_{1/n}$.

Finally, we fix $z \in G$ and define G_n as the component of $\widetilde{G}_{1/n}$ containing z. Then $G_n \subset G_{n+1} \subset G$ and $\bigcup_n G_n = G$; indeed, let $x \in G$, take a curve $\gamma \subset G$ joining x and z, then $\gamma \subset \widetilde{G}_{1/n_1} \cup \cdots \cup \widetilde{G}_{1/n_m} = \widetilde{G}_{1/\max n_k}$ and $x \in \gamma \subset G_{\max n_k}$. □

Let $D \subset \mathbb{R}^3$ and $G = S_D$. The construction of the objects in the proof of the above result shows that the sets $S_{A(\mathcal{P}(G_n))}$, where $\mathcal{P} : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ is the projection, are strongly pseudoconvex domains exhausting the domain $S_{A(D)}$ for any isometry of $\mathbb{R}^3$. Thus Theorem 1 follows from the same result for the strongly pseudoconvex case contained in [1]. However, it seems to us that the proof of Theorem 1 presented by us is simpler and more self-contained.

3 More problems related to semitube domains

Note that the reasoning used in the proof of Theorem 1 also implies the following property of pseudoconvex Hartogs–Laurent and semitube domains.

Proposition 4. *Let $G \subset \mathbb{C}^2$ be a pseudoconvex Hartogs–Laurent domain with the base $\Omega \subset \mathbb{C}$. Consider the function*

$$t : \Omega \ni z \longmapsto \text{number of components of } G_z,$$

where $G_z := G \cap (\{z\} \times \mathbb{C})$. Then t is lower semicontinuous.

Consequently, if $D \subset \mathbb{R}^3$ is such that S_D is a pseudoconvex semitube domain, then the function

$$s : D_1 \ni z \longmapsto \text{number of components of } D \cap (\{z\} \times \mathbb{R}),$$

where $D_1 := \{z \in \mathbb{C} : D \cap (\{z\} \times \mathbb{R}) \neq \emptyset\}$, is lower semicontinuous.

Proof. Fix $z_0 \in \Omega$. Let $w_1, \ldots, w_k \in G_{z_0}$ be points from different components of G_{z_0}. Using the Kontinuitätssatz for the annuli (as in the proof of the previous theorem) we easily get that for $z \in \Omega$ sufficiently close to z_0 the number of components of G_z is at least k, which finishes the proof. The case of semitube domains follows from the case of Hartogs–Laurent domains by applying the result for the domain $\pi(S_D)$. □

Note that the above property easily implies that the semitube domain over the torus in a 'vertical position' (and many others) as described in Section 6.4 of [1] is not pseudoconvex.

In view of Theorem 1 it would also be interesting and natural to consider the following problem. Let $D \subset \mathbb{C}^n$ be a domain satisfying the following condition: for every real isometry A of $\mathbb{C}^n = \mathbb{R}^{2n}$ the set $A(D)$ is pseudoconvex. Does it follow that D is convex? Certainly the problem is non-trivial for $n \geq 2$. We show now that the answer is negative for $n \geq 2$, too.

Proposition 5. *Let $n \geq 2$. Then there is a non-convex domain $D \subset \mathbb{C}^n$ such that $A(D)$ is pseudoconvex for every real isometry of $\mathbb{C}^n = \mathbb{R}^{2n}$.*

Proof. First we consider a class of functions defined on domains $\Omega \subset \mathbb{R}^m$ with $m \geq 2$. We call an upper semicontinuous function $u : \Omega \longrightarrow [-\infty, \infty)$ *multisubharmonic* if u restricted to $\Omega \cap (L + a)$ is subharmonic for every two-dimensional subspace $L \subset \mathbb{R}^m$ and a point $a \in \mathbb{R}^m$ such that $\Omega \cap (L + a) \neq \emptyset$. Let us make the last statement precise: the function u on $\Omega \cap (L + a)$ is considered to be subharmonic if for some (any) pair of vectors X and Y forming an orthonormal basis of L the function $(t, s) \longmapsto u(a + tX + sY)$ is subharmonic on its domain (lying in $\mathbb{R}^2$). Certainly, in the case of u being $\mathcal{C}^2$ we have the following simple description:

$$\Delta_{X,Y} u(a) := \frac{\partial^2 u}{\partial X^2}(a) + \frac{\partial^2 u}{\partial Y^2}(a) \geq 0$$

for $X, Y \in \mathbb{R}^m$, $||X|| = ||Y|| = 1$, $\langle X, Y \rangle = 0$ and $a \in \Omega$. It is clear that any multisubharmonic function (in $\mathbb{C}^n = \mathbb{R}^{2n}$) is plurisubharmonic and that these two concepts are the same in $\mathbb{C}$.

For $m \geq 2$ and $\alpha \in (0, 1]$ consider the following function

$$u(x) := \frac{1}{2}(x_1^2 + \cdots + x_{m-1}^2 - \alpha x_m^2).$$

We have

$$\Delta_{X,Y} u(a) = X_1^2 + \cdots + X_{m-1}^2 - \alpha X_m^2 + Y_1^2 + \cdots + Y_{m-1}^2 - \alpha Y_m^2.$$

Then for every orthonormal pair X, Y we get $\Delta_{X,Y} u(a) = 2 - (1 + \alpha)(X_m^2 + Y_m^2)$. Note that

$$(1 - X_m^2)(1 - Y_m^2) = (X_1^2 + \cdots + X_{m-1}^2)(Y_1^2 + \cdots + Y_{m-1}^2) \geq (X_1 Y_1 + \cdots + X_{m-1} Y_{m-1})^2 = X_m^2 Y_m^2,$$

whence $X_m^2 + Y_m^2 \leq 1$ and $\Delta_{X,Y} u(a) \geq 1 - \alpha$, so u is multisubharmonic.

Now we define the set

$$D := \{z \in \mathbb{C}^n : u(z) < 1\} \quad (m := 2n).$$

Note that D is connected and non-convex. It follows from the multisubharmonicity of u that $A(D)$ is pseudoconvex for every real isometry A. □

Funding: This work is partially supported by grants of the Polish National Science Centre no. UMO-2011/03/B/ST1/04758 and DEC-2012/05/N/ST1/03067.

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Received 31 July, 2013.